If $a = b = \phi$, then $= \phi$ and the equation $\sqrt[3]{x+\phi} = \sqrt[3]{x-\phi} + \sqrt[3]{\phi}$ with $x > \phi$, has the solution

$$x = \frac{3\sqrt{3}\left(\phi - \phi\right) + 4\phi\sqrt{7}}{6\sqrt{3}} = \frac{2\sqrt{21}}{9}\phi.$$

Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Ashland University Undergraduate Problem Solving Group, Ashland, OH; D. M. Bătinetu-Giurgiu of "Matei Basarab" National College, Bucharest, Romania with **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania; Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; the proposer, and Students from Taylor University (see below);**

Students at Taylor University, Upland, IN.
Group 1. Ben Byrd, Maddi Guillaume, and Makayla Schultz.
Group 2. Caleb Knuth, Michelle Franch and Savannah Porter.
Group 3. Lauren Moreland, Anna Souzis, and Boni Hermandez

- **5404:** *Proposed Arkady Alt, San Jose, CA*

For any given positive integer $n \geq 3$, find the smallest value of the product of $x_1 x_2 \ldots x_n$, where $x_1, x_2, x_3, \ldots x_n > 0$ and $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \ldots + \dfrac{1}{1+x_n} = 1.$

Solution 1 by Ed Gray, Highland Beach, FL

Suppose each term had the value of $\dfrac{1}{n}$. Since there are n terms, the sum is equal to 1, satisfying the problem restriction.

In the event for each $k, 1 \leq k \leq n$

1. $\dfrac{1}{1+x_k} = \dfrac{1}{n}$, so $x_k = n-1$, and the value of the product is:
2. $(n-1)^n$.

If this is not the smallest product, at least one value of x_k must be less than $n-1$. Suppose $x_k = n - 1 - e$ where $e > 0$.

Then the series contains the therm $\dfrac{1}{1+x_k} = \dfrac{1}{n-e}$. We must increase the value of another term so that the sum maintains the value of 1. We must have:

3. $\dfrac{1}{n-e} + \dfrac{1}{1+x_m} = \dfrac{2}{n}$
4. $\dfrac{1}{1+x_m} - \dfrac{2}{n} - \dfrac{1}{n-e} = \dfrac{2(n-e-n)}{n(n-e)} = \dfrac{2n-2e-n}{n(n-e)}$
5. $\dfrac{1}{1+x_m} = \dfrac{n-2e}{n(n-e)}$
6. $(1+x_m)(n-2e) = n(n-e)$
7. $1 + x_m = \dfrac{n(n-e)}{n-2e}$

8. $x_m = \dfrac{n(n-e)}{n-2e} - 1 = \dfrac{n(n-e)-n-2e)}{n-2e} = \dfrac{n^2-ne-n+2e}{n-2e}$

9. The new product is: $\left((n-1)^{n-2}\right) x_k x_m$. If the new product is to be smaller, we must have:

10. $\dfrac{(n-1)^{n-2}(n-1-e)(n^2-n-e(n-2)}{n-2e} < (n-1)^n$, or dividing by $(n-1)^{n-2}$

11. $(n-1-e)(n^2-n-en+2e) < (n-2e)(n-1)^2$,

12. $(n-1-e)(n^2-n-en+=2e) < (n-2e)(n^2-2n+1)$, which simplifies to:

13. $2en^2+ne2 < 2e^2$. Dividing by e^2,

14. $\dfrac{2n^2}{e}+n < 2$, which is a contraction. Therefore, we did not decrease the product, but increased it.

So $(n-1)^n$ is the minimum product.

Solution 2 by Ramya Dutta (student), Chennai Mathematical Institute) India

Consider the polynomial $P(x) = \prod_{j=1}^{n}(x+x_j)$, then $\dfrac{P'(x)}{P(x)} = \sum_{j=1}^{n}\dfrac{1}{x+x_j}$, i.e., $P'(1) = P(1)$.
Denoting the j-th symmetric polynomial by, $\sigma_j = \sum_{1\le k_1<k_2<\cdots<k_j\le n} x_{k_1}x_{k_2}\cdots x_{k_j}$ for $j \ge 1$ and $\sigma_0 = 1$,
$P(x) = \sum_{j=0}^{n}\sigma_j x^{n-j}$ and $P'(x) = \sum_{j=0}^{n-1}(n-j)\sigma_j x^{n-j-1}$
Therefore, the condition $P(1) = P'(1)$ is equivalent to,

$$\sigma_n = \sum_{j=0}^{n-1}(n-j-1)\sigma_j$$

Using, AM-GM inequality: $\sigma_j \ge \binom{n}{j}\sigma_n^{j/n}$ for $j \ge 1$.
I.e., writing $\sigma_n^{1/n} = \alpha$, we have,

$$\begin{aligned}\alpha^n = \sum_{j=0}^{n-1}(n-j-1)\sigma_j &\ge \sum_{j=0}^{n-1}(n-j-1)\binom{n}{j}\alpha^j\\ &= (n-1)\sum_{j=0}^{n-1}\binom{n}{j}\alpha^j - n\sum_{j=1}^{n-1}\binom{n-1}{j-1}\alpha^j\\ &= (n-1)\left((1+\alpha)^n-\alpha^n\right) - n\alpha\left((1+\alpha)^{n-1}-\alpha^{n-1}\right)\\ &= \alpha^n-(1+\alpha)^n+n(1+\alpha)^{n-1}\end{aligned}$$

that is, $(1+\alpha)^n \ge n(1+\alpha)^{n-1} \implies \alpha \ge n-1$ (since, $\alpha > 0$)
So, the minimum value of $x_1x_2\cdots x_n$ is $(n-1)^n$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We shall use the Method of Lagrange Multipliers to show that the smallest value of the product is $(n-1)^n$, achieved when each $x_i = n-1$.

First suppose that all but one of the x_i are equal: let $x_i = b$ for $1 \le i \le n-1$ and choose x_n so that the constraint $\displaystyle\sum_{i=1}^{n} \frac{1}{1+x_i} = \frac{1}{1+x_1} + \frac{1}{1+x_2} \ldots + \frac{1}{1+x_n} = 1$ is satisfied:

$$\sum_{i=1}^{n} \frac{1}{1+x_i} = (n-1)\frac{1}{1+b} + \frac{1}{1+x_n} = 1, \Longrightarrow x_n = \frac{n-1}{b-(n-2)}, \text{ where}$$

$b > n-2$ to make $x_n > 0$.

Then the product $f(x_1, x_2, \ldots, x_)n) = \displaystyle\prod_{i=1}^{n} x_i = b^{n-1} \frac{n-1}{b-(n-2)}$.

We note that as b becomes unbounded positive, the product of the $x_i's$ becomes unbounded positive, and as b approaches $n-2$ from above, the product of the $x_i's$ also becomes unbounded positive. Thus if the product has an absolute extremum subject to the given constraint, it must be a minimum since the product is unbounded above.

For $b = n-1$, we see that $\mathrm{x}_n = n-1$, so every $x_i = n-1$ and the product is equal to $(n-1)^n$,

We consider this as a Lagrange Multiplier problem where we minimize the product $f(x_1, x_2, \ldots, x_n) = \displaystyle\prod_{i=1}^{n} x_i$ subject to the constraint

$$\sum_{i=1}^{n} \frac{1}{1+x_i} = \frac{1}{1+x_1} + \frac{1}{1+x_2} \ldots + \frac{1}{1+x_n} = 1.$$

That is, subject to the constraint

$$g(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{1+x_i} = \frac{1}{1+x_1} + \frac{1}{1+x_2} \ldots + \frac{1}{1+x_n} = 1.$$

By the Method of Lagrange Multipliers, we'll find the minimum of f where $\displaystyle\frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_n) = \lambda \frac{\partial}{\partial x_i} g(x_1, x_2, \ldots, x_n)$ for $1 \le k \le n$.

We see that: $\displaystyle\frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_n) = \prod_{\substack{i=1 \\ i=k}}^{n} x_i$ and $\displaystyle\frac{\partial}{\partial x_i} g(x_1, x_2, \ldots, x_n) = \frac{1}{(1+x_i)^2}$ for $1 \le k \le n$.

Thus we want to solve the system, $\displaystyle\prod_{\substack{i=1 \\ i=k}}^{n} x_i = \frac{\lambda}{(1+x_k)^2}$, for $1 \le k \le n$.

Solving each equation for λ gives $\lambda = -(1+x_k)^2 \displaystyle\prod_{\substack{i=1 \\ i\neq k}}^{n} x_i$ for $1 \le k \le n$.

Hence, for any $1 \le j,\ k \le n$ we must have $\lambda = -(1+x_i)^2 \displaystyle\prod_{\substack{i=1 \\ i\neq k}}^{n} x_i = -(1+x_j)^2 \prod_{\substack{i=1 \\ i\neq k}}^{n} x_i$

Algebra gives $\dfrac{x_j}{(1+x_j)^2} = \dfrac{x_k}{(1+x_k)^2}, \quad 1 \leq j,\ k \leq n.$

We claim this forces $x_i = x_k$. Suppose that $x_k \neq x_i$ for some $k \neq j$.

Now consider the function $h(x) = \dfrac{x}{(1+x)^2}$ for $x > 0$.

Note that $h(x_i) = h(x_k)$ for $1 \leq j,\ k \leq n$

By calculus, $h(x)$ is strictly increasing for $0 < x < 1$ to a maximum (of $1/4$) at $x = 1$, and is then strictly decreasing for $x > 1$. That is, h except for the peak at $x = 1$ is two- to- one function (for $x > 0$).

Moreover, $h(x)$ has the reflective property $h\left(\dfrac{1}{x}\right) = h(x)$. Hence, for $1 \leq j \neq k \leq n,\ h(x_j) = h(x_k)$ and $x_j \neq x_k \Longrightarrow x_j = \dfrac{1}{x_k}$. Then jour constraint becomes

$$\begin{aligned}
1 &= \frac{1}{1+x_k} + \frac{1}{1+x_j} + \text{(other positive terms)} \\
&= \frac{1}{1+x_k} + \frac{1}{1+\dfrac{1}{x_k}} + \text{(other positive terms)} \\
&= \frac{1}{1+x_k} + \frac{x_k}{1+x_k} + \text{(other positive terms)} \\
&= 1 + \text{(other positive terms)}
\end{aligned}$$

which is impossible. Therefore, $x_k = x_j$.

Hence, to achieve the extreme value, which must be a minimum, all of the x_i are equal and must equal $n-1$, forcing the minimum value of the product to be $(n-1)^n$.

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Denote by $\dfrac{1}{1+x_i} = y_i \Longrightarrow x_i = \dfrac{1-y_i}{y_i},\ y_i > 0, i = 1, 2, , n$

By the AM-GM, we get

$$x_1x_2\ldots x_n = \prod_{i=1}^{n} \frac{1-y_i}{y_i} = \frac{y_2+y_3+\ldots+y_n}{y_1} \ldots \frac{y_1+y_2+\ldots+y_{n-1}}{y_n} \geq \frac{(n-1)^n \sqrt[n-1]{(y_1y_2\ldots y_n)^{n-1}}}{y_1y_2\ldots y_n} = (n-1)^n.$$

So, $x_1x_2\ldots x_n \geq (n-1)^n$. Equality occurs for $x_1 = x_2 = \ldots = x_n = n-1$.

Editor's comment : In addition to a general solution to this problem, the problem's author, **Arkady Alt of San Jose, CA,** also provided 4 different solutions for the cases $n = 2 = 3$.

Solution A.

Let $n = 3$. We have $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \dfrac{1}{1+x_3} = 1 \Longleftrightarrow$
$3 + 2(x_1 + x_2 + x_3) + x_1x_2 + x_2x_3 + x_3x_1 = 1 + x_1 + x_2 + x_3 + x_1x_2 + x_2x_3 + x_3x_1 + x_1x_2x_3 \iff 2 + x_1 + x_2 + x_3 = x_1x_2x_3$. Since $x_1 + x_2 + x_3 \geq 3\sqrt[3]{x_1x_2x_3}$

then $x_1x_2x_3 \geq 2 + 3\sqrt[3]{x_1x_2x_3} \iff \left(\sqrt[3]{x_1x_2x_3} - 2\right)\left(\sqrt[3]{x_1x_2x_3} + 1\right)^2 \geq 0 \iff$ $\sqrt[3]{x_1x_2x_3} - 2 \geq 0 \iff x_1x_2x_3 \geq 2^3$.

Solution B.

Since $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \dfrac{1}{1+x_3} = 1 \iff \dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} = \dfrac{x_3}{1+x_3} \iff$ $\dfrac{1+x_3}{1+x_1} + \dfrac{1+x_3}{1+x_2} = x_3 \implies x_3 \geq 2(1+x_3)\sqrt{\dfrac{1}{1+x_1} . \dfrac{1}{1+x_2}} = \dfrac{2(1+x_3)}{\sqrt{(1+x_1)(1+x_2)}}$.

Similarly we obtain $x_2 \geq \dfrac{2(1+x_2)}{\sqrt{(1+x_3)(1+x_1)}}, x_1 \geq \dfrac{2(1+x_1)}{\sqrt{(1+x_2)(1+x_3)}}$.

Hence, $x_1x_2x_3 \geq \dfrac{2^3(1+x_1)(1+x_2)(1+x_3)}{\sqrt{(1+x_2)(1+x_3)} \cdot \sqrt{(1+x_3)(1+x_1)} \cdot \sqrt{(1+x_1)(1+x_2)}} = 2^3$.

Solution C.

Let $a := \dfrac{1}{1+x_1}, b := \dfrac{1}{1+x_2}, c := \dfrac{1}{1+x_3}$ then $a, b, c \in (0,1), a+b+c = 1$ and $x_1 = \dfrac{1-a}{a} = \dfrac{b+c}{a} \geq \dfrac{2\sqrt{bc}}{a}, x_2 = \dfrac{1-b}{b} = \dfrac{c+a}{b} \geq \dfrac{2\sqrt{ca}}{b}, x_3 = \dfrac{1-c}{c} = \dfrac{a+b}{c} \geq \dfrac{2\sqrt{ab}}{c}$.

Therefore, $x_1x_2x_3 \geq \dfrac{2\sqrt{bc}}{a} \cdot \dfrac{2\sqrt{ca}}{b} \cdot \dfrac{2\sqrt{ab}}{c} = 8$.

Solution D.

First note that at least one of the products x_1x_2, x_2x_3, x_3x_1 must be greater then 1.

Indeed,assume that $x_1x_2, x_2x_3, x_3x_1 \leq 1$. Then since $2 + x_1 + x_2 + x_3 = x_1x_2x_3 \iff$ $1 = \dfrac{2}{x_1x_2x_3} + \dfrac{1}{x_1x_2} + \dfrac{1}{x_2x_3} + \dfrac{1}{x_3x_1}$ and $x_1x_2x_3 = \sqrt{x_1x_2 \cdot x_2x_3 \cdot x_3x_1} \leq 1$

we obtain a contradiction $1 = \dfrac{2}{x_1x_2x_3} + \dfrac{1}{x_1x_2} + \dfrac{1}{x_2x_3} + \dfrac{1}{x_3x_1} \geq 2 + 1 + 1 + 1 \geq 5$.

Let it be $x_1x_2 > 1$ and let $t := \sqrt{x_1x_2}, r := x_1x_2x_3$.

Then $2 + x_1 + x_2 + x_3 = x_1x_2x_3$ becomes

$+\dfrac{r}{t^2} = r$ and, since $x_1 + x_2 \geq 2\sqrt{x_1x_2} = 2t,\ t > 1$, we obtain

$$r - \frac{r}{t^2} = 2 + x_1 + x_2 \geq 2 + 2t \iff \frac{r\left(t^2-1\right)}{t^2} \geq 2(t+1) \iff r \geq \frac{2t^2}{t-1} = 2\left(\frac{t^2-1+1}{t-1}\right) =$$
$$2\left(\left(t - 1 + \frac{1}{t-1}\right) + 2\right) \geq 2(2+2) = 8, \text{ because } t - 1 + \frac{1}{t-1} \geq 2.$$

Comment by Editor: **Neculai Stanciu of "George Emil Palade" School, Buzău, Romania and Titu Zvonaru of Comănesti, Romania,** stated that there is a paper in the Romanian Mathematical Gazette, (Volume CXX, number 11, 2015) pp. 489-498 by **Eugen Păltănea** that presents five solutions and extensions for the following proposition: Let $x_1, x_2, \ldots, x_n > 0, n \geq 2$. If $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \ldots + \dfrac{1}{1+x_n} = 1$, then $\sqrt[n]{x_1x_2\ldots x_n} \geq n-1$. They presented a new solution to this proposition and then applied it to problem 5404.

Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg,